When the Morse index is infinite

Alberto Abbondandolo and Pietro Majer

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Abstract

Let f be a smooth Morse function on an infinite dimensional separable Hilbert manifold, all of whose critical points have infinite Morse index and co-index. For any critical point x choose an integer $a(x)$ arbitrarily. Then there exists a Riemannian structure on M such that the corresponding gradient flow of f has the following property: for any pair of critical points x, y, the unstable manifold of x and the stable manifold of y have a transverse intersection of dimension $a(x) - a(y)$.

Introduction

Let f be a smooth Morse function on an infinite dimensional separable Hilbert manifold[1] M. Let us denote by $\mathrm{crit}(f)$ the set of its critical points, and let us assume that each $x \in \mathrm{crit}(f)$ has finite Morse index $i(x)$. A Riemannian structure g on M determines the vector field $-\mathrm{grad}\, f$, whose local flow $\phi_t : M \to M$ has the critical points of f as rest points. The Morse condition is translated into the fact that these rest points are hyperbolic. Their unstable and stable manifolds,

$$W^u(x; f, g) = W^u(x) = \left\{ p \in M \mid \lim_{t \to -\infty} \phi_t(p) = x \right\},$$
$$W^s(x; f, g) = W^s(x) = \left\{ p \in M \mid \lim_{t \to +\infty} \phi_t(p) = x \right\},$$

are submanifolds with $\dim W^u(x) = \mathrm{codim}\, W^s(x) = i(x)$. Moreover for a generic choice of the Riemannian structure g, $W^u(x)$ and $W^s(y)$ intersect transversally for every pair of critical points x, y. It follows that $W^u(x) \cap W^s(y)$ - if non-empty - is a submanifold of dimension

$$\dim W^u(x) \cap W^s(y) = i(x) - i(y). \tag{1}$$

If the critical points have infinite Morse index and co-index, their unstable and stable manifolds are infinite dimensional and so could be their intersections. However, there are situations in which these intersections are indeed finite dimensional, and one can associate an integer $i_{\mathrm{rel}}(x)$ to each critical point x in such a way that

$$\dim W^u(x) \cap W^s(y) = i_{\mathrm{rel}}(x) - i_{\mathrm{rel}}(y). \tag{2}$$

This fact allows to develop the analogue of Morse theory for such functions: it is the case of Floer homology[2] for the Hamiltonian action functional on the space of loops on certain symplectic manifolds [Flo88a, Flo88b, Flo89], of functionals on Hilbert spaces somehow compatible with either a fixed splitting into two closed linear subspaces [AM01], or with a closed linear subspace

[1] By this we mean a paracompact separable space with a smooth atlas of charts taking values in an infinite dimensional separable real Hilbert space.

[2] Actually, in Floer homology one does not consider the gradient flow with respect to a Riemannian metric on a Hilbert manifold, but one uses the gradient equation with respect to an inner product which is not complete on the space where the function is smooth. Therefore, one does not obtain a local flow and there are no stable and unstable manifolds. However, the space of solutions connecting two critical points is a finite dimensional manifold.

and a fixed flag of finite dimensional linear subspaces [Szu92, KS97, GIP99], and more generally of functionals on Hilbert manifolds which are compatible with a fixed subbundle of the tangent bundle [AM03] or with a polarization [CJS95].

The aim of this note is to prove the following result:

THEOREM. *Let f be a smooth Morse function on the Hilbert manifold M, all of whose critical points have infinite Morse index and co-index. Let*

$$a : \mathrm{crit}(f) \to \mathbb{Z}$$

be an arbitrary *function. Then there exists a Riemannian structure g on M such that for every $x, y \in \mathrm{crit}(f)$ the intersection $W^u(x; f, g) \cap W^s(y; f, g)$ is transverse and - if it is non-empty - it is a submanifold of dimension $a(x) - a(y)$.*
Moreover, if $\{(x_i, y_i) \mid i = 1, \dots, n\}$ is a finite set of pairs of critical points such that $a(x_i) > a(y_i)$ and there is a smooth curve $u_i : [0, 1] \to M$, $u_i(0) = x_i$, $u_i(1) = y_i$, $i = 1, \dots, n$, such that $Df(u_i)[u_i'] < 0$ on $]0, 1[$, the Riemannian structure g can be chosen in such a way that $W^u(x_i; f, g) \cap W^s(y_i; f, g) \neq \emptyset$ for every $i = 1, \dots, n$.
Finally, if g_0 is any Riemannian structure on M, the Riemannian structure g can be chosen to be equivalent to g_0, meaning that there is $c > 0$ for which

$$\frac{1}{c}\, g_0(\xi, \xi) \leq g(\xi, \xi) \leq c\, g_0(\xi, \xi) \quad \forall \xi \in TM.$$

Therefore the situation is completely different from the case of finite Morse indices, where the identity (1) holds for every choice of the Riemannian structure and Morse theory only depends on the pair (M, f). In the case of infinite Morse indices and co-indices no Morse theory based just on the pair (M, f) can possibly exist, and extra structures, such as the ones used in the above mentioned papers, are really needed.

Notice that if the function a is chosen to be constant, the above theorem produces a Riemannian structure g for which the unstable and stable manifolds of two distinct critical points never meet. The statement about the possibility of requiring some intersections to be non-empty says that choosing a non-constant a one can actually produce non-trivial intersections.

Finally, the possibility of choosing g to be equivalent to a given Riemannian structure says that from the point of view of completeness and compactness the Riemannian structure g is not worse than a preferred Riemannian structure g_0 we might dispose of: if (M, g_0) is complete so is (M, g), if f satisfies the Palais-Smale condition with respect to g_0 it will also satisfy it with respect to g.

The idea of the proof is the following. After choosing a suitable Riemannian structure in a neighborhood of the critical points, we can find an integrable subbundle $\mathcal{V}$ of TM having infinite dimension and codimension and such that for every critical point x the negative eigenspace of the Hessian of f at x is a *compact perturbation* of $\mathcal{V}(x)$, of *relative dimension* $a(x)$. The existence of $\mathcal{V}$ follows from a well known result of Eells and Elworthy [EE70] stating in particular that every separable infinite dimensional Hilbert manifold can be smoothly embedded as an open subset of a Hilbert space (hence its tangent bundle has many integrable subbundles) and from Kuiper theorem [Kui65] stating that the general linear group of an infinite dimensional Hilbert space is contractible. The fact that $\mathcal{V}$ is integrable allows us to construct a vector field X on M having f as a Lyapunov function, and whose local flow *essentially preserves* $\mathcal{V}$. As it is shown in [AM03], these facts imply that the unstable and stable manifolds of two rest points x, y of the vector field X have a *Fredholm intersection* of index $a(x) - a(y)$. The conclusion follows from the fact that X is the negative gradient of f with respect to a suitable Riemannian structure on M, which can be perturbed in order to have transverse intersections.

1 Flows on a Hilbert space which essentially preserve a closed linear subspace

Let H be the infinite dimensional separable real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. The operator norm will be denoted by $\|\cdot\|$. The set of all closed linear subspaces of H will

be denoted by $Gr(H)$, and $Gr_{\infty,\infty}(H)$ will denote the subset of all those subspaces having infinite dimension and codimension.

We start by recalling some basic definitions and some results from [AM03]. A pair (V, W) of closed linear subspaces of H is said a *Fredholm pair* if $\dim V \cap W < \infty$ and $\operatorname{codim}(V + W) < \infty$, in which case the number $\operatorname{ind}(V, W) = \dim V \cap W - \operatorname{codim}(V + W)$ is said the *Fredholm index* of (V, W). Two submanifolds of H have a *Fredholm intersection* if at every point of their intersection the pair consisting of their tangent spaces is Fredholm.

Given V, W two closed linear subspaces of the Hilbert space H, V is said to be a *compact perturbation* of W if the difference of the orthogonal projections onto V and onto W is a compact operator, or equivalently if the operators

$$P_V P_{W^\perp} \quad \text{and} \quad P_W P_{V^\perp} \tag{3}$$

are compact. In this case, $(V, W^\perp)$ is a Fredholm pair and its index is called the *relative dimension* of V with respect to W, and it is denoted by $\dim(V, W)$.

Let M be an open subset of H, and let X be a smooth vector field on M, having only hyperbolic rest points (this means that the differential of X at every rest point does not have purely imaginary spectrum). Let W be a closed linear subspace of H, with orthogonal projector $P = P_W$, and consider the following compatibility conditions between X and W:

(C1) For every rest point x of X, the positive eigenspace of the differential of X at x is a compact perturbation of W.

(C2) For every $u \in H$, the operator $[DX(u), P]P = (I - P)DX(u)P$ is compact.

By (C1), the relative dimension of the positive eigenspace of $DX(x)$ with respect to W is a well defined integer, which is said the *relative Morse index* of x, denoted by $i(x, W)$. If ϕ_t denotes the local flow determined by the vector field X, condition (C2) is equivalent to the fact that ϕ_t *essentially preserves* W, in the sense that $D\phi_t(u)W$ is a compact perturbation of W, for every (t, u) in the domain of the local flow.

Lemma 1.1 *The set of smooth vector fields which satisfy (C2) with respect to W is a module over $C^\infty(M, \mathbb{R})$.*

Indeed, the last term in the identity

$$[D(\varphi X)(u), P]P = \varphi(u)[DX(u), P]P + D\varphi(u)[P\cdot](I - P)X(u)$$

has rank one. The importance of conditions (C1) and (C2) lies in the following fact:

Proposition 1.2 ([AM03], Theorem 1.6) *Assume that the vector field X has only hyperbolic rest points and satisfies (C1), (C2). Then for every pair of rest points x, y, the immersed submanifolds $W^u(x)$ and $W^s(y)$ have Fredholm intersection of index $i(x, W) - i(y, W)$.*

When the Fredholm index of these intersections is positive, the vector field can be locally perturbed so as to make these intersections transversal at some point. For instance, we have the following:

Proposition 1.3 ([AM04]) *Let x, y be hyperbolic rest points of a smooth vector field $X : M \to H$ which satisfies (C1) and (C2). Let $u_0 \in W^u(x; X) \cap W^s(y; X)$ and let u be the integral line of X such that $u(0) = u_0$. If $i(x, W) > i(y, W)$, for every $\epsilon > 0$ and every $k \in \mathbb{N}$ there exists a smooth vector field $Y : M \to H$ such that:*

(i) Y satisfies (C1) and (C2);

(ii) $Y = X$ on $M \setminus B_\epsilon(u_0)$ and on $u(\mathbb{R})$;

(iii) $\|Y - X\|_{C^k} < \epsilon$;

(iv) $W^u(x;Y)$ *and* $W^s(y;Y)$ *meet transversally at* u_0.

If the vector field X is the negative gradient of a Morse function f with respect to some Riemannian structure g_0 on M, the rest points of X are exactly the critical points of f, and the unstable and the stable manifolds of these points are embedded submanifolds. In this case, we can perturb the Riemannian structure g_0 in such a way that the new negative gradient flow of f has the unstable and stable manifolds of pairs of critical points intersecting transversally:

PROPOSITION 1.4 ([AM04]) *Assume that f is a smooth Morse function on the open subset $M \subset H$ endowed with the Riemannian structure g_0, such that the vector field $-\mathrm{grad}_{g_0} f$ satisfies (C1) and (C2) with respect to a closed linear subspace W. Let $k \in \mathbb{N}$. Then there exists a C^k-dense set of smooth Riemannian structures g, equivalent to g_0, such that $-\mathrm{grad}_g f$ satisfies (C1) and (C2) with respect to W and the intersections $W^u(x; f, g) \cap W^s(y; f, g)$ are transverse, for every pair of critical points x, y.*

2 Preliminary lemmata

We recall that the support of a diffeomorphism ϕ is the closure of the set where $\phi \neq \mathrm{id}$.

LEMMA 2.1 *Let $A_0 \in GL(H)$, $r_1 > 0$. Then there exists a number $r_0 \in]0, r_1[$ and a smooth diffeomorphism $\phi : H \to H$ with support in the ball of radius r_1, such that $\phi(u) = A_0 u$ for $|u| \leq r_0$.*

Proof. By Kuiper theorem [Kui65], the general linear group $GL(H)$ is connected, so we can find a curve $A \in C^\infty(\mathbb{R}, GL(H))$ such that $A(t) = A_0$ for $t \leq r_1/2$ and $A(t) = I$ for $t \geq r_1$. Since the continuous function $t \mapsto \|A(t)^{-1}A'(t)\|$ has support in $]0, r_1]$, it is easy to find a number $r_0 \in]0, r_1[$ and a smooth function $\mu :]0, +\infty[\to]0, +\infty[$ such that $\mu(t) = t$ for $t \leq r_0$ and $t \geq r_1$, and

$$\mu'(t) > \|A(t)^{-1}A'(t)\|\mu(t) \quad \forall t > 0. \tag{4}$$

The map $\phi : H \to H$ defined by

$$\phi(x) = \begin{cases} \frac{1}{|x|}\mu(|x|)A(|x|)x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

is clearly smooth, coincides with $x \mapsto A_0 x$ for $|x| \leq r_0$ and with the identity for $|x| \geq r_1$.

We claim that ϕ is bijective. Indeed, if $v \in H$, $v \neq 0$, the equation $\phi(x) = v$ is equivalent to

$$x = tu, \quad \mu(t)A(t)u = v,$$

where $t > 0$ and $|u| = 1$. By (4),

$$\begin{aligned} \frac{d}{dt}\left(\mu(t)^{-2}|A(t)^{-1}v|^2\right) &= -2\mu(t)^{-2}\left(\frac{\mu'(t)}{\mu(t)}|A(t)^{-1}v|^2 + \langle A(t)^{-1}v, A(t)^{-1}A'(t)A(t)^{-1}v\rangle\right) \\ &\leq -2\mu(t)^{-2}\left(\frac{\mu'(t)}{\mu(t)} - \|A(t)^{-1}A'(t)\|\right)|A(t)^{-1}v|^2 < 0, \end{aligned}$$

so the function $t \mapsto \mu(t)^{-1}|A(t)^{-1}v|$ is strictly decreasing. Moreover this function maps $]0, +\infty[$ onto $]0, +\infty[$. Therefore there exists a unique $t > 0$ such that $\mu(t)^{-1}|A(t)^{-1}v| = 1$, and $x = tu = t\mu(t)^{-1}A(t)^{-1}v$ is the unique solution of $\phi(x) = v$.

We claim that ϕ is a diffeomorphism. By the inverse mapping theorem it is enough to show that $D\phi(x)$ is invertible for every $x \in H$. This is certainly true if $x = 0$, and if $x = tu$ with $t > 0$ and $|u| = 1$ we have

$$D\phi(x)[v] = \frac{\mu(t)}{t}A(t)v \quad \forall v \in \langle u\rangle^\perp.$$

Therefore $D\phi(x)$ is invertible if and only if the vector $\partial_t \phi(tu)|_{t=|x|}$ does not belong to the hyperplane

$$D\phi(x)\langle u\rangle^\perp = A(t)\langle u\rangle^\perp = \langle A(t)^{*-1}u\rangle^\perp.$$

By (4) the quantity

$$\langle \partial_t \phi(tu), A(t)^{*-1}u\rangle = \langle (\mu' A + \mu A')u, A^{*-1}u\rangle = \langle (\mu' + \mu A^{-1}A')u, u\rangle \geq \mu' - \mu \|A^{-1}A'\|$$

is strictly positive for every $t > 0$, hence $\partial_t \phi(tu)|_{t=|x|}$ does not belong to the hyperplane $D\phi(x)\langle u\rangle^\perp$. □

The following result is a simple addendum to the Morse Lemma:

LEMMA 2.2 *Let $f : H \to \mathbb{R}$ be a smooth function such that $f(0) = 0$, $Df(0) = 0$, and $D^2 f(0) = A$ is invertible. For every $r_1 > 0$ there exists a smooth diffeomorphism $\phi : H \to H$ with support in the ball of radius r_1, such that $\phi(0) = 0$, $D\phi(0) = I$, and*

$$f(u) = \frac{1}{2}\langle A\phi(u), \phi(u)\rangle, \tag{5}$$

for $|u| \leq r_0$.

Indeed, Palais' proof of the Morse Lemma [Pal63] produces a local diffeomorphism at 0 ϕ verifying $\phi(0) = 0$, $D\phi(0) = I$, and (5). Let $\chi : \mathbb{R} \to$ be a smooth function with compact support and such that $\chi = 1$ in a neighborhood of 0. Therefore for $\epsilon > 0$ small,

$$\phi_\epsilon(u) = u + \chi\left(\frac{|u|}{\epsilon}\right)(\phi(u) - u)$$

defines a smooth map on H, which coincides with ϕ in a neighborhood of 0, hence satisfying (5) therein and $D\phi_\epsilon(0) = I$, and which coincides with the identity for $|u| \geq r_1$. Since

$$\lim_{\epsilon \to 0} \|D\phi_\epsilon - I\|_\infty = 0,$$

ϕ_ϵ is a global diffeomorphism for ϵ small.

The Banach manifolds embedding theorem of Eells and Elworthy [EE70] implies that every infinite dimensional separable Hilbert manifold has a smooth open embedding into the Hilbert space H. Here we can ask that this embedding satisfies also some more conditions:

LEMMA 2.3 *Let (M, g_0) be a Riemannian Hilbert manifold, let $f : M \to \mathbb{R}$ be a smooth Morse function whose critical points have infinite Morse index and co-index. Let $V : \mathrm{crit}(f) \to Gr_{\infty,\infty}(H)$ be an arbitrary function. Then there exists a smooth open embedding $\psi : M \to H$ such that for every $x \in \mathrm{crit}(f)$:*

(i) $D\psi(x) : T_x M \to H$ is an isometry;

(ii) there exists $r(x) > 0$ and a self-adjoint operator $A(x) \in GL(H)$ whose negative eigenspace is $V(x)$, and such that

$$f(\psi^{-1}(\psi(x) + u)) = f(x) + \frac{1}{2}\langle A(x)u, u\rangle,$$

for every $u \in B_{r(x)}(0)$.

Proof. Let $\psi_0 : M \to H$ be an open embedding, as given by Eells and Elworthy's theorem [EE70]. For every $x \in \mathrm{crit}(f)$ let $T(x) \in GL(H)$ be such that $T(x)D\psi_0(x)$ is an isometry from $(T_x M, g_0)$ to H, and such that $T(x)D\psi_0(x)$ maps the negative eigenspace of the g_0-Hessian of f at x onto $V(x)$. The possibility of finding such a $T(x)$ follows from the fact that the unitary group of H acts transitively on the component $Gr_{\infty,\infty}(H)$ of the Grassmannian of H. By Lemma 2.1,

there exists a smooth diffeomorphism $\phi_1 : H \to H$ supported in a neighborhood of $\psi_0(\mathrm{crit}(f))$ such that $\phi_1(\psi_0(x) + u) = \psi_0(x) + T(x)u$ for every $x \in \mathrm{crit}(f)$ and $|u|$ small. Then $f_1 = f \circ \psi_0^{-1} \circ \phi_1^{-1}$ is a smooth Morse function on an open subset of H, and the negative eigenspace of $A(x) = D^2 f_1(\phi_1(\psi_0(x)))$ is $V(x)$, for every $x \in \mathrm{crit}(f)$. By Lemma 2.2, we can find a smooth diffeomorphism $\phi_2 : H \to H$ supported in a neighborhood of $\mathrm{crit}(f_1)$ such that for every $x \in \mathrm{crit}(f_1)$ there holds $\phi_2(x) = x$, $D\phi_2(x) = I$, and

$$f_1(u) = f_1(x) + \frac{1}{2}\langle A(x)\phi_2(u), \phi_2(u)\rangle,$$

for every u close to x. The smooth open embedding $\phi_2^{-1} \circ \phi_1 \circ \psi_0$ satisfies all the requirements. □

3 Proof of the theorem

Fix some $W \in Gr_{\infty,\infty}(H)$. By Lemma 2.3 we may assume that M is an open subset of H, that the Riemannian structure g_0 coincides with $\langle\cdot,\cdot\rangle$ at every critical point of f, and that every $x \in \mathrm{crit}(f)$ has a neighborhood $U_x \subset M$ such that

$$f(u) = f(x) + \frac{1}{2}\langle A(x)(u-x), (u-x)\rangle \quad \forall u \in U_x, \tag{6}$$

where the negative eigenspace $V^-(A(x))$ of the invertible self-adjoint operator $A(x) = D^2 f(x)$ is a compact perturbation of W with $\dim(V^-(A(x)), W) = a(x)$. Up to replacing g_0 by a uniformly equivalent metric and up to replacing the neighborhoods U_x by smaller ones, we may assume that g_0 coincides with $\langle\cdot,\cdot\rangle$ on the whole U_x, for every $x \in \mathrm{crit}(f)$.

We denote by $|\cdot|_{g_0(u)}$ the norm associated to the inner product given by g_0 at u, $|v|^2_{g_0(u)} = g_0(u)[v,v]$, and by $\|\cdot\|_{g_0(u)}$ the corresponding operator norm on the space of bounded linear operators on H,

$$\|L\|_{g_0(u)} = \sup_{\substack{v\in H \\ |v|_{g_0(u)}=1}} |Lv|_{g_0(u)}.$$

We will define a vector field X on M, having f as a Lyapunov function, and such that the unstable and stable manifolds of its rest points have the required properties. Then we will find a Riemannian structure g_1 for which $X = -\mathrm{grad}_{g_1} f$. Finally we will perturb g_1 in order to have a Morse-Smale flow.

The first step of the proof consists in modifying the curves u_i in order to make them integral lines in the intersections $W^u(x_i) \cap W^s(y_i)$, with respect to the vector field X we are defining.

LEMMA 3.1 *There exist $\theta_0 \in [0,1[$, $T > 0$ and for every $i = 1, \dots, n$ there are curves $v_i \in C^\infty(\mathbb{R}, M) \cap C^0(\overline{\mathbb{R}}, M)$ such that $v_i(\mathbb{R}) \cap v_j(\mathbb{R}) = \emptyset$ if $i \neq j$,*

$$v_i(-\infty) = x_i, \quad v_i(+\infty) = y_i, \quad v_i'(t) = -\mathrm{grad}_{g_0} f(v_i(t)) \text{ for } |t| \geq T,$$
$$v_i([-\infty, -T]) \subset U_{x_i}, \quad v_i([T, +\infty]) \subset U_{y_i},$$

and

$$|v_i'(t) + \mathrm{grad}_{g_0} f(v_i(t))|_{g_0(v_i(t))} \leq \theta_0 |\mathrm{grad}_{g_0} f(v_i(t))|_{g_0(v_i(t))} \quad \forall t \in \mathbb{R}. \tag{7}$$

Proof. By modifying the curve $u_i : [0,1] \to M$ in a neighborhood of 0 and 1, we can join it with two negative gradient flow orbits in the unstable manifold of x_i and in the stable manifold of y_i, constructing a curve $\tilde{v}_i \in C^\infty(\mathbb{R}, M) \cap C^0(\overline{\mathbb{R}}, M)$ satisfying all the requirements, with the possible exception of (7), which will be achieved by a reparameterization.

The construction of the curve $\tilde{v}_i$ is the following, where we omit the index i in order to simplify the notation. Let $r > 0$ be such that $B_r(x) \subset U_x$ and $B_r(y) \subset U_y$, so that f has the quadratic form (6) on $B_r(x)$ and on $B_r(y)$, and

$$\mathrm{grad}_{g_0} f(\xi) = A(x)(\xi - x) \quad \forall \xi \in B_r(x), \quad \mathrm{grad}_{g_0} f(\xi) = A(y)(\xi - y) \quad \forall \xi \in B_r(y).$$

Let $\epsilon > 0$ be such that $u(\epsilon) \in B_r(x)$ and $u(1-\epsilon) \in B_r(y)$. Since the level sets $\Sigma^- = \{\xi \in B_r(x) \mid f(\xi) = f(u(\epsilon))\}$ and $\Sigma^+ = \{\xi \in B_r(y) \mid f(\xi) = f(u(1-\epsilon))\}$ are connected and meet $x + V^-(A(x))$, respectively $y + V^+(A(y))$, we can find a continuous piecewise smooth curve $w : \mathbb{R} \to M$ such that $w(t) = u(t)$ for $\epsilon \leq t \leq 1-\epsilon$, $w(t) \in \Sigma^-$ for $0 \leq t \leq \epsilon$, $w(t) \in \Sigma^+$ for $1-\epsilon \leq t \leq 1$, $w(0) \in x + V^-(A(x))$, $w(1) \in y + V^+(A(y))$, and

$$w(t) = x + e^{-tA(x)}(w(0) - x) \quad \forall t < 0, \quad w(t) = y + e^{-(t-1)A(y)}(w(1) - y) \quad \forall t > 1.$$

The sets Σ^- and Σ^+ are leafs of the codimension one foliation given by the level sets of f. Therefore, it is easy to modify w in a neighborhood of $[0, \epsilon]$ and $[1-\epsilon, 1]$, obtaining a curve $\tilde{v}$ with the required properties. By slightly perturbing these curves, we may assume that the supports of $\tilde{v}_i$ are pairwise disjoint.

Let τ be a solution of the ODE

$$\tau' = -\frac{\langle \mathrm{grad}_{g_0} f(\tilde{v}(\tau)), \tilde{v}'(\tau)\rangle_{g_0(\tilde{v}(\tau))}}{|\tilde{v}'(\tau)|^2_{g_0(\tilde{v}(\tau))}}.$$

Then $\tau'(t) > 0$ for every $t \in \mathbb{R}$, and $\tau' = 1$ for $|t|$ large, so $\tau : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, and it is a constant shift for t in a neighborhood of $-\infty$ and of $+\infty$. Setting $v(t) = \tilde{v}(\tau(t))$, we have that

$$\begin{aligned} |v' + \mathrm{grad}_{g_0} f(v)|_{g_0(v)} &= \left| -\frac{\langle \mathrm{grad}_{g_0} f(\tilde{v}(\tau)), \tilde{v}'(\tau)\rangle_{g_0(\tilde{v}(\tau))}}{|\tilde{v}'(\tau)|^2_{g_0(\tilde{v}(\tau))}} \tilde{v}'(\tau) + \mathrm{grad}_{g_0} f(\tilde{v}(\tau)) \right| \\ &= \min_{\lambda \in \mathbb{R}} |\lambda \tilde{v}'(\tau) + \mathrm{grad}_{g_0} f(\tilde{v}(\tau))|_{g_0(\tilde{v}(\tau))} = |\mathrm{grad}_{g_0} f(v)|_{g_0(v)} \sin\alpha, \end{aligned}$$

where $\alpha = \alpha(t)$ is the angle between the vector $\tilde{v}'(\tau(t))$ and $-\mathrm{grad}_{g_0}(\tilde{v}(\tau(t)))$, with respect to the inner product $g_0(\tilde{v}(\tau(t)))$. By the properties of $\tilde{v}$, the continuous function α takes values in $[0, \pi/2[$ and it vanishes for $|t|$ large. Hence α is bounded away from $\pi/2$, and (7) holds with $\theta_0 = \max \sin\alpha < 1$. □

The second step consists in defining the vector field X in a neighborhood of the support of the curves v_i:

Lemma 3.2 *Let $\theta \in]\theta_0, 1[$. There exist an open neighborhood U_* of $\bigcup_{i=1}^n v_i(\overline{\mathbb{R}})$ and a smooth vector field $X_* : U_* \to H$ such that:*

(i) X_* satisfies (C1) and (C2) with respect to W;

(ii) $X_*(u) = -\mathrm{grad}_{g_0} f(u)$ in a neighborhood of $\{x_1, \dots, x_n, y_1, \dots, y_n\}$;

(iii) $v_i'(t) = X_*(v_i(t))$ for every $t \in \mathbb{R}$ and every $i = 1, \dots, n$;

(iv) $|X_*(u) + \mathrm{grad}_{g_0} f(u)|_{g_0(u)} \leq \theta |\mathrm{grad} f(u)|_{g_0(u)}$ for every $u \in U_*$;

(v) $W^u(x_i; X_*)$ and $W^s(y_i; X_*)$ intersect transversally along the flow line v_i, for every $i = 1, \dots, n$.

Proof. We start by defining a smooth vector field $\tilde{X}$ on a neighborhood $\tilde{U}$ of $\bigcup_{i=1}^n v_i(\overline{\mathbb{R}})$ satisfying (i), (ii), (iii). For every $i = 1, \dots, n$, let $U_i : [-2T, 2T] \to U(H) \cap (I + \mathcal{L}_c(H))$ be a smooth path in the group of unitary operators on H which are compact perturbations of the identity, such that

$$U_i(t)\langle v_i'(0)\rangle^\perp = \langle v_i'(t)\rangle^\perp \quad \forall t \in [-2T, 2T].$$

For instance, if $P_i(t)$ denotes the orthogonal projection onto the one-dimensional linear subspace $< v_i'(t) >$, one can choose U_i to be the solution of the linear Cauchy problem

$$\begin{cases} U_i'(t) = [P_i'(t), P_i(t)] U_i(t), \\ U_i(0) = I. \end{cases}$$

See [Kat80], section II - §4.5. If $r > 0$ is small enough, the map

$$\psi_i :]-2T, 2T[\times (\langle v_i'(0)\rangle^\perp \cap B_r(0)) \to H, \quad (t,\xi) \mapsto v_i(t) + U_i(t)\xi,$$

is a diffeomorphism onto a tubular neighborhood V_i of $v_i(]-2T,2T[)$. By choosing a smaller r, we may also assume that $V_i \cap \mathrm{crit}(f) = \emptyset$ and that $V_i \cap V_j = \emptyset$ if $i \neq j$. The smooth vector field $\psi_{i*}(\partial/\partial t)$,

$$\psi_{i*}\left(\frac{\partial}{\partial t}\right)(u) = D\psi_i(\psi_i^{-1}(u))[(1,0)] \quad \forall u \in V_i,$$

has $v_i|_{]-2T,2T[}$ as an integral flow line and satisfies (C2) with respect W, as its differential at every point is a compact operator. Let $V_0 \subset \bigcup_{i=1}^n U_{x_i} \cup U_{y_i}$ be an open neighborhood $\bigcup_{i=1}^n v_i(\overline{\mathbb{R}} \setminus]-2T,2T[)$ such that $v_i^{-1}(V_0) \cap [-T,T] = \emptyset$ for every $i = 1,\dots,n$. Let $\{\chi_i\}_{i=0}^n$ be a smooth partition of unity subordinated to the open covering $\{V_i\}_{i=0}^n$ of $\tilde{U} := V_0 \cup V_1 \cup \dots \cup V_n$. By Lemma 1.1, the smooth vector field

$$\tilde{X} : \tilde{U} \to H, \quad \tilde{X} = -\chi_0 \mathrm{grad}_{g_0} f + \sum_{i=1}^n \chi_i \psi_{i*}\left(\frac{\partial}{\partial t}\right),$$

satisfies (i), (ii), (iii). By Proposition 1.3, we can modify $\tilde{X}$ in a small neighborhood of the set $\{v_1(0),\dots,v_n(0)\}$, obtaining a smooth vector field $X_* : \tilde{U} \to H$ which satisfies also (v). By (ii), (iii), and (7), X_* satisfies also (iv) in a smaller neighborhood $U_* \subset \tilde{U}$ of $\bigcup_{i=1}^n v_i(\overline{\mathbb{R}})$. □

The third step consists in defining the vector field X in a neighborhood of the critical points other than x_i and y_i. Let $x \in \mathrm{crit}(f) \setminus \bigcup_{i=1}^n \{x_i, y_i\}$. We define X_x to be the affine vector field

$$X_x(u) = -A(x)(u - x).$$

Its unique rest point is x, and the positive eigenspace of $DX_x(x)$ is $V^+(DX_x(x)) = V^+(-A(x)) = V^-(A(x))$, which was chosen to be a compact perturbation of W, so X_x satisfies (C1) with respect to W, and the relative Morse index of x is $i(x, W) = a(x)$. Moreover by (3),

$$[DX_x(u), P]P = -[A(x), P]P = (P-I)A(x)P = (P-I)A(x)P_{V^+(A(x))}P - (I-P)P_{V^-(A(x))}A(x)P$$

is compact, so X_x satisfies (C2) with respect to W. Since g_0 coincides with $\langle\cdot,\cdot\rangle$ on U_x, by (6) we have

$$X_x(u) = -\mathrm{grad}_{g_0} f(u) \quad \forall u \in U_x. \tag{8}$$

The fourth step consists in defining the vector field X in a neighborhood of all the remaining points. We shall denote by M_0 the complement of $\bigcup_{i=1}^n v_i(\overline{\mathbb{R}})$ in M. Let $x \in M_0 \setminus \mathrm{crit}(f)$, and consider the constant vector field

$$X_x(u) = -\mathrm{grad}_{g_0} f(x),$$

which trivially satisfies (C2) with respect to W. Let $U_x \subset M_0 \setminus \mathrm{crit}(f)$ be an open neighborhood of x such that

$$|X_x(u) + \mathrm{grad}_{g_0} f(u)|_{g_0(u)} \leq \theta\, |\mathrm{grad}_{g_0} f(u)|_{g_0(u)} \quad \forall u \in U_x. \tag{9}$$

Now we can patch the vector fields X_x, $x \in M_0$, and X_* by a partition of unity. Indeed, let $\{\varphi_j\}_{j\in J}$ be a smooth partition of unity subordinated to a locally finite refinement of the open covering $\{U_x\}_{x \in M_0 \cup \{*\}}$ of M: φ_j is non-negative, it has support in $U_{\omega(j)}$, $\omega(j) \in M_0 \cup \{*\}$, every point of M has a neighborhood which intersects finitely many supports of φ_j's, and $\sum_{j\in J} \varphi_j = 1$. Consider the smooth vector field on M:

$$X(x) = \sum_{j \in J} \varphi_j(x) X_{\omega(j)}(x).$$

By Lemma 1.1, X satisfies (C2) with respect to W. From (8) and from the fact that the condition in Lemma 3.2 (iv) and (9) are convex conditions, we deduce

$$|X(u) + \mathrm{grad}_{g_0} f(u)|_{g_0(u)} \leq \theta \, |\mathrm{grad}_{g_0} f(u)|_{g_0(u)} \quad \forall u \in M. \tag{10}$$

In particular, the set of rest points of X is $\mathrm{crit}(f)$. By Lemma 3.2 (ii) and (8),

$$X(u) = -\mathrm{grad}_{g_0} f(u) \quad \forall x \in U'_x, \ \forall x \in \mathrm{crit}(f), \tag{11}$$

for some neighborhood U'_x of the critical point x. So $DX(x) = -A(x)$ for every critical point x, hence X satisfies (C1) with respect to W, and the relative Morse index of x is

$$i(x, W) = \dim(V^-(A(x)), W) = a(x). \tag{12}$$

By Lemma 3.2 (iii) and (v), for every $i = 1, \dots, n$ the unstable manifold $W^u(x_i; X)$ meets the stable manifold $W^s(y_i; X)$ transversally along the flow line v_i (indeed, by construction $v_i(t) \in \operatorname{supp} \varphi_j$ if and only if $\omega(j) = *$).

We claim that there exists a Riemannian structure g_1 on M, uniformly equivalent to g_0, such that $X = -\mathrm{grad}_{g_1} f$. The construction of g_1 makes use of the following lemma, whose straightforward proof is left to the reader:

Lemma 3.3 *Let $x \in M$ and $u, v \in H$ be such that $g_0(u, v)_x > 0$. Then the bounded linear operator $L_x(u, v)$ defined by*

$$L_x(u, v)w = w - \frac{g_0(w, u)_x}{|u|^2_{g_0(x)}} u + \frac{g_0(w, v)_x}{g_0(u, v)_x} v, \quad w \in H,$$

is $g_0(x)$-self-adjoint, $g_0(x)$-positive, invertible, satisfies

$$L_x(u, v)u = v, \quad L_x(u, u) = I,$$

and

$$\begin{aligned} \|L_x(u, v)\|_{g_0(x)} &\leq 1 + \frac{|v|^2_{g_0(x)}}{g_0(u, v)_x}, \\ \|L_x(u, v)^{-1}\|_{g_0(x)} &\leq \left(1 + \frac{|u|_{g_0(x)} |v|_{g_0(x)}}{g_0(u, v)_x}\right)^2 + \frac{|u|^2_{g_0(x)}}{g_0(u, v)_x}. \end{aligned} \tag{13}$$

Every Riemannian structure g_1 on M can be written as

$$g_1(u, v)_x = g_0(G(x)u, v)_x, \quad \forall x \in M,$$

for some smooth map $G : M \to GL(H)$, such that $G(x)$ is $g_0(x)$-symmetric and $g_0(x)$-positive for every x. In this case, the g_1-gradient of f is related to the g_0-gradient of f by $\mathrm{grad}_{g_0} f = G \, \mathrm{grad}_{g_1} f$. Moreover, g_1 is uniformly equivalent to g_0 if and only if

$$\sup_{x \in M} \|G(x)\|_{g_0(x)} < +\infty, \quad \sup_{x \in M} \|G(x)^{-1}\|_{g_0(x)} < +\infty.$$

Then we can define a Riemannian structure g_1 by setting

$$G(x) = \begin{cases} I & \text{for } x \in \bigcup_{y \in \mathrm{crit}(f)} U'_y, \\ L_x(X(x), -\mathrm{grad}_{g_0} f(x)) & \text{for } x \in M \setminus \mathrm{crit}(f). \end{cases}$$

Indeed, the above formula defines a smooth map because the map $(x, u, v) \mapsto L_x(u, v)$ is smooth on

$$\{(x, u, v) \in M \times H \times H \mid g_0(u, v)_x > 0\},$$

and because of (11) and $L_x(u,u) = I$. Since $L_x(u,v)u = v$, we obtain for $x \in M \setminus \mathrm{crit}(f)$,

$$G(x)X(x) = L_x(X(x), -\mathrm{grad}_{g_0} f(x))X(x) = -\mathrm{grad}_{g_0} f(x) = -G(x)\mathrm{grad}_{g_1} f(x),$$

from which $X = -\mathrm{grad}_{g_1} f$ on M. Finally, an easy computation shows that (10) and (13) imply

$$\|G(x)\|_{g_0(x)} \leq 1 + \frac{1}{1-\theta}, \quad \|G(x)^{-1}\|_{g_0(x)} \leq \frac{4}{(1-\theta)^2} + \frac{(1+\theta)^2}{1-\theta}, \quad \forall x \in M,$$

so g_1 is uniformly equivalent to g_0.

By Proposition 1.4, we can perturb g_1 obtaining a Riemannian structure g on M such that the vector field $-\mathrm{grad}_g f$ still satisfies (C1) and (C2) with respect to W, all the intersections between unstable and stable manifolds are transverse, and the transverse intersections at v_i are preserved, for every $i = 1, \dots, n$. Then Proposition 1.2 and (12) imply that

$$\dim W^u(x; f, g) \cap W^s(y; f, g) = i(x, W) - i(y, W) = a(x) - a(y),$$

whenever such intersection is non-empty.

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